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Lacunary Quadrature Formulae and Interpolation Singularity*

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Birkhoff quadrature formulae (q.f.), which have algebraic degree of precision (ADP) greater than the number of values used, are studied. In particular, we construct a class of quadrature rules of ADP $=2n+2r+1$ which are based on the information $\{f^{(j)}(-1), f^{(j)}(1), j=0, ..., r-1; f(x_i), f^{(2m)}(x_i), i=1, ..., n\}$, where m is a positive integer and $r=m$, or $r=m-1$. It is shown that the corresponding Birkhoff interpolation problems of the same type are not regular at the quadrature nodes. This means that the constructed quadrature formulae are not of interpolatory type. Finally, for each m, we prove the existence of a quadrature formula based on the information $\{f(x_i), f^{(2m)}(x_i), i=1, ..., 2m\}$, which has algebraic degree of precision $4m+1$. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let $E=(e_{ij})_{i=1,\ j=0}^{n,\ N}$ be a given incidence matrix, (i.e., E contains only 0 and 1 entries). Denote by $|E|$ the number of 1's in E. Matrix E defines a class Ω of q.f. of the type

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{e_{ij}=1} a_{ij} f^{(j)}(x_i), \tag{1}$$

with real coefficients $\{a_{ij}\}$ and nodes $\mathbf{X} \in \Xi$, where Ξ is the simplex $\{\mathbf{X}=(x_1, ..., x_n)\colon -1 \leqslant x_1 < \cdots < x_n \leqslant 1\}$. The purpose of this paper is to construct q.f. (1) which have ADP greater than $|E|-1$. Our study is based on certain basic facts from Birkhoff interpolation (see [5]). Let us recall some of them. The pair $(E, \mathbf{X})$, $\mathbf{X} \in \Xi$, defines the so-called Birkhoff interpolation problem

$$P_f \in \pi_N : P_f^{(j)}(x_i) = f^{(j)}(x_i) \qquad \text{for} \quad e_{ij}=1$$

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(π_N denotes the set of polynomials of degree $\leqslant N$). The pair $(E, \mathbf{X})$, $\mathbf{X} \in \Xi$, is said to be *regular* if the corresponding interpolation problem has a unique solution for each sufficiently smooth f. Otherwise $(E, \mathbf{X})$ is called *singular*. The following is an immediate observation (see [5, Theorem 10.2(ii)]).

THEOREM A. *Let $(E, \mathbf{X})$ be a regular pair. Then a quadrature formula $Q \in \mathfrak{Q}$ is exact for every polynomial of degree $|E| - 1$ if and only if this coincides with the corresponding interpolatory quadrature formula.*

We call the matrix E poised if $(E, \mathbf{X})$ is regular for each $\mathbf{X}$. It is well known that if $(E, \mathbf{X})$ is almost poised, then the set of points $\mathbf{X}$ in Ξ where $(E, \mathbf{X})$ is singular is a nowhere-dense set of measure 0 (see [5, p. 5]). We call the quadrature rules of the form

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} (\lambda_i f(x_i) + \delta_i f^{(m)}(x_i))$$

$(0, m)$ quadrature formulae with n nodes. Turán [12, Problem XXXIII] asked whether there exists a $(0, 2)$ q.f. with n nodes, which is exact for all $f \in \pi_{2n}$. The problem was solved by the author in [3]. We extend here the question of Turán for q.f. of more general Birkhoff type. There are only a few papers in the literature [1, 3, 7, 13–15] dealing with Birkhoff quadrature formulae of high algebraic degree of precision. Explicit formulae are constructed mostly for cases in which the nodes are located at the zeros $\mathbf{X}_n^{(\alpha)}$ of the classical ultraspherical polynomials $P_n^{(\alpha)} := P_n^{(\alpha,\alpha)}$, orthogonal on $[-1, 1]$ with respect to $\omega_\alpha(x) = (1 - x^2)^\alpha$, $\alpha \in (-1, \infty)$. Varma [13] solved problems XXXVI–XXXIX of [12], providing a q.f. of the form

$$\int_{-1}^{1} f(x)\,w(x)\,dx \approx \sum_{j=0}^{r-1} (a_j f^{(j)}(-1) + b_j f^{(j)}(1)) + \sum_{i=1}^{n} (\lambda_i f(x_i) + \delta_i f^{(m)}(x_i)) \tag{2}$$

with $w = 1$, $m = 2$, $r = 1$, n even, set of nodes $\mathbf{X}_n^{(1)}$, and $\mathrm{ADP} = 2n + 3$. Later on, Varma [14] found a very nice simple way to obtain this q.f. for arbitrary $n \in \mathbb{N}$. Nevai and Varma [7] gave a q.f. (2) with $w = \omega_\alpha$, $m = 2$, $r = 1$, set of nodes $\mathbf{X}_n^{(\alpha)}$, $\mathrm{ADP} = 2n + 1$, and Varma and Saxena [15] did the same for $w = 1$, $m = 3$, $m = 4$, $r = 2$, nodes $\mathbf{X}_n^{(1)}$, and $\mathrm{ADP} = 2n + 1$. Akhlaghi, Chak, and Sharma [1] found a $(0, 3)$ q.f. with nodes $\mathbf{X}_{n-2}^{(1)} \cup \{-1, 1\}$, $n \geqslant 3$, and $\mathrm{ADP} = 2n - 1$.

According to Theorem A, $\mathrm{ADP}(Q) \geqslant |E| - 1$ for interpolatory q.f. On the other other hand, only the formulae given in [13, 14, 3] satisfy

$\mathrm{ADP}(Q) \geq |E|$. We construct a class of quadrature rules of Birkhoff type (2) which have ADP greater than $|E|$. The main result is the following.

THEOREM 1. *Let $m, n \in \mathbb{N}$ and $\{x_i\}_1^n$ be the zeros of $P_n^{(m)}$. Then there exist quadrature formulae*

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{j=0}^{m-1} d_j(f^{(j)}(-1) + (-1)^j f^{(j)}(1)) + \sum_{i=1}^{n} (\gamma_i f(x_i) + \delta_i f^{(2m)}(x_i)), \quad (3)$$

with $\mathrm{ADP} = 2n + 2m + 1$ *and*

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{j=0}^{m-2} d_j(f^{(j)}(-1) + (-1)^j f^{(j)}(1)) + \sum_{i=1}^{n} (\gamma_i f(x_i) + \delta_i f^{(2m)}(x_i)), \quad (4)$$

with $\mathrm{ADP} = 2n + 2m - 1$, *respectively.*

Note that (4) reduces to a q.f. of type (0, 2) in the case $m = 1$.

The interpolation pairs $(E, \mathbf{X})$ are singular at the nodes $\mathbf{X}$ of the constructed formulae.

THEOREM 2. *Let E be the $(n+2) \times (2n+2m)$ incidence matrix, associated with the q.f. (3), and let $\mathbf{X} = (-1, x_1, ..., x_n, 1)$ be the set of nodes of (3). Then the interpolation pair $(E, \mathbf{X})$ is singular.*

We prove also

THEOREM 3. *Let $m \in \mathbb{N}$ and $\{x_i\}_1^{2m}$ be the zeros of $P_{2m}^{(m)}$. Then the interpolatory quadrature formula*

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{2m} (\lambda_i f(x_i) + \mu_i f^{(2m)}(x_i)) =: Q_{0,2m}(f) \quad (5)$$

is exact for every polynomial of degree $4m + 1$.

2. PRELIMINARIES

As usual $P_n^{(\alpha)}$ will be normalized by

$$P_n^{(\alpha)}(1) = \binom{n+\alpha}{n}. \quad (6)$$

We need the relations (see [10, (4.3.1), (4.21.7)])

$$(1-x^2)^\alpha P_n^{(\alpha)}(x)=\frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n [(1-x^2)^{\alpha+n}], \tag{7}$$

$$\frac{d}{dx} P_n^{(\alpha)}(x)=\frac{n+2\alpha+1}{2} P_{n-1}^{(\alpha+1)}(x). \tag{8}$$

It follows immediately from (8) and (6) that

$$\left(\frac{d}{dx}\right)^\nu P_n^{(\alpha)}(1)=2^{-\nu}\frac{(n+2\alpha+\nu)!}{(n+2\alpha)!}\binom{n+\alpha}{n-\nu},$$
$$\binom{n+\alpha}{n-\nu}:=0 \qquad \text{for} \quad \nu>n. \tag{9}$$

LEMMA 1. *For arbitrary $n, L, k \in \mathbb{N}$ with $k<L$ we have*

$$\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{(n+2L+\nu)!}{(L-k+\nu)!\,(L+\nu)!}(n-\nu+1)_\nu$$
$$=\frac{(n+2L)!}{L!\,(L+k)!}\frac{(n+L+k)!}{(n+L-k)!}, \tag{10}$$

where $(c)_\nu := c(c+1)\cdots(c+\nu-1)$ for $\nu>0$ and $(c)_0=1$.

Proof. The sum on the left-hand side of (10) is equal to

$$\frac{(n+2L)!}{L!\,(L-k)!}\sum_{\nu=0}^{k}\frac{(-k)_\nu(n+2L+1)_\nu(-n)_\nu}{\nu!\,(L-k+1)_\nu(L+1)_\nu}$$
$$=\frac{(n+2L)!}{L!\,(L-k)!}\cdot {}_3F_2\left(\begin{matrix}-k, & -n, & n+2L+1; \\ & L+1, & L-k+1;\end{matrix}\ 1\right).$$

An application of Saalschutz' formula [4, Chap. 2.1, formula (30)] gives the identity (10). ∎

3. PROOFS

Let us consider the so-called "Generalized Lobatto quadrature formulae," i.e., formulae of the form

$$\int_{-1}^{1} f(x)\,\omega_\alpha(x)\,dx \approx \sum_{j=0}^{L-1}(a_j f^{(j)}(-1)+b_j f^{(j)}(1))+\sum_{i=1}^{n}\lambda_i f(x_i)$$
$$:=Q_1(f;\alpha,L), \tag{11}$$

where $a_j := a_j(\alpha, L, n)$, $b_j := b_j(\alpha, L, n)$, $\lambda_j := \lambda_j(\alpha, L, n)$, and $L \in \mathbb{N} \cup \{0\}$ (when $L=0$ the first sum disappears). There exists a unique formula of form (11) with ADP $= 2n+2L-1$, because (11) is a Gaussian quadrature rule with preassigned nodes (such nodes here are -1 and 1), whose existence and unicity have been proved by Stancu [9]. For $L=0$, (11) reduces to Gaussian q.f. with respect to ω_α. The nodes $\{x_i\}_1^n$ are located at $\mathbf{X}_n^{(\alpha+L)}$ and $a_j = (-1)^j b_j$. An application of (11) to $f(x) = g(x)\,\omega_L(x)$ with $g \in \pi_{2n-1}$ yields

$$\lambda_i = \lambda_{in}^{(\alpha+L)}/\omega_L(x_i),$$

where $\lambda_{in}^{(\alpha)}$ are the Cotes numbers of the Gaussian quadrature formula with the weight ω_α (see [10, p. 352, (15.3.1)]). Maskell and Sack [6] proposed a method for the evaluation of the coefficients relative to the endpoints even for a more general case, treating integrals with Jacobi weights. However, this generalization complicates the calculations and does not yield explicit formulae, except for the coefficients $b_{L-1}(\alpha, L, n)$ and $b_{L-2}(\alpha, L, n)$. In our particular case $(\alpha = 0)$ we find closed form expressions for the coefficients.

LEMMA 2. *If $\alpha = 0$ then the coefficients b_j of Q_1 are given by the recurrence relation*

$$\begin{aligned} b_{L-1} &= \frac{I_{L,n,L-1}}{D_{L,n,L-1}^{(L-1)}} \\ b_j &= \frac{1}{D_{Lnj}^{(j)}} \left\{ I_{Lnj} - \sum_{\nu=j+1}^{L-1} D_{Lnj}^{(\nu)} b_j \right\}, \qquad j = L-2, \ldots, 0. \end{aligned} \tag{12}$$

with

$$I_{Lnj} = \frac{2^{j+L+1}}{j+1} \frac{\binom{n+L-j-1}{n}}{\binom{n+L+j+1}{n+L}}, \qquad j < L \tag{13}$$

$$D_{Lnj}^{(j+k)} = (-1)^j 2^{L-k} (j+k)! \binom{n+L+k}{n} \binom{n+L}{k}, \qquad k, j < L, \tag{14}$$

Proof. Note that (11) holds for polynomials

$$g_j(x) := (1-x)^j (1+x)^L P_n^{(L)}(x), \qquad j = 0, 1, \ldots, L-1.$$

If we set $I_{Lnj} := \int_{-1}^{1} g_j(x)\, dx$ and $D_{Lnj}^{(j+k)} := g_j^{(j+k)}(1)$ then $b_{L-1}(0, L, n)$, $b_{L-2}(0, L, n)$, ..., $b_0(0, L, n)$ could be obtained successively by recurrence relation (12). Let us note that (13) is an immediate consequence of (3.7) in [6]. Thus we need only prove (14). By using the Leibniz rule twice, (7) and (9), we obtain

$$D_{Lnj}^{(j+k)}=(-1)^j\frac{(j+k)!}{k!}\,[(1+x)^L\,P_n^{(L)}(x)]^{(k)}\,(1)$$

$$=(-1)^j\frac{(j+k)!}{k!}\sum_{v=0}^{k}\binom{k}{v}\frac{L!}{(L-k+v)!}\,2^{L-k+v}$$

$$\times\left(\frac{d}{dx}\right)^v P_n^{(L)}(1)$$

$$=(-1)^j\,2^{L-k}\,\frac{(j+k)!}{n!\,k!}\,\frac{L!\,(n+L)!}{(n+2L)!}$$

$$\times\sum_{v=0}^{k}\binom{k}{v}\frac{(n+2L+v)!}{(L-k+v)!\,(L+v)!}\,(n-v+1)_v,$$

which together with (10) gives (14). ∎

It is easily seen from (12), (13), and (14) that

$$b_{L-1}=(-1)^{L-1}\,2^L\,\frac{n!\,L!}{(n+2L)!}$$

$$\text{for}\quad L\geqslant 1,$$

$$b_{L-2}=(-1)^{L-2}\,2^{L-1}\,\frac{n!\,L!}{(n+2L)!}\,\frac{L}{L+1}\,\{2n^2+2(1+2L)\,n+(1+L)^2\}$$

$$\text{for}\quad L\geqslant 2$$

$$b_{L-3}=(-1)^{L-3}\,2^{L-3}\,\frac{n!\,L!}{(n+2L)!}\,\frac{1}{(L+1)^2\,(L+2)}$$

$$\times\{(n+1)(n+2)(L+1)^2\,(L+2)-(L-2)(L-1)(L+1)$$

$$\times(n+L-1)_4+2(L-2)\,L(L+2)(n+L)$$

$$\times(n+L+1)\{2n^2+2(1+2L)\,n+(1+L)^2\}\}$$

$$\text{for}\quad L\geqslant 3.$$

LEMMA 3. *Let $m, n\in\mathbb{N}$, $L\in\mathbb{N}\cup\{0\}$. The quadrature formula*

$$\int_{-1}^{1} f(x)\,dx\approx\sum_{j=0}^{m-1} c_j(f^{(j)}(-1)+(-1)^{(j)}\,f^{(j)}(1))$$
$$+\frac{(-1)^m}{(2m)!}\,Q_1(f^{(2m)};m,L) \tag{15}$$
$$:=Q_2(f;m,L),$$

where

$$c_j = c_j(m) := (2^{j+1}/(j+1)!) \cdot \binom{m}{j+1} \Big/ \binom{2m}{j+1}$$

is exact for every polynomial of degree $2n+2L+2m-1$.

Proof. The following identity is known as "Tchakaloff–Obrechkoff quadrature formula" [11, 8].

$$\int_{-1}^{1} f(x)\,dx = \sum_{j=0}^{m-1} c_j (f^{(j)}(-1) + (-1)^j f^{(j)}(1)) + \frac{(-1)^m}{(2m)!} \int_{-1}^{1} (1-x^2)^m f^{(2m)}(x)\,dx. \tag{16}$$

Applying $Q_1(f; m, L)$ to the integral on the right-hand side of (16) we obtain (15) and see that $Q_2(f; m, L)$ is exact for every $f \in \pi_{2(n+L+m)-1}$. ▮

The following lemma is a consequence of an application of (11) and (15).

Lemma 4. *For every* $\xi \in \mathbb{R}$, $m, n \in \mathbb{N}$, $L \in \mathbb{N} \cup \{0\}$ *the quadrature formula*

$$Q(f; \xi; m, L) := \xi \cdot Q_2(f; m, L) + (1-\xi)\, Q_1(f; 0, m+L)$$

has the form

$$Q(f; \xi; m, L) = \sum_{j=0}^{m+L-1} d_j (f^{(j)}(-1) + (-1)^j f^{(j)}(1)) + \sum_{j=2m}^{2m+L-1} d_j (f^{(j)}(-1) + (-1)^j f^{(j)}(1)) + \sum_{i=1}^{n} (\gamma_i f(x_i) + \delta_i f^{(2m)}(x_i)).$$

The nodes $\{x_i\}_1^n$ *are located at the zeros of* $P_n^{(m+L)}$. *Moreover, it is exact for every polynomial of degree* $2(n+m+L)-1$.

In the proofs of Theorems 1 *and* 2 *we shall mean by* $\{x_i\}_1^n$ *the zeros of* $P_n^{(m)}$.

Proof of Theorem 1. Let us apply Lemma 4 for $L=0$. For every $m, n \in \mathbb{N}$, $Q(f; \xi; m, 0)$ takes the form

$$Q(f; \xi; m, L) = \sum_{j=0}^{m-1} d_j (f^{(j)}(-1) + (-1)^j f^{(j)}(1)) + \sum_{i=1}^{n} (\gamma_i f(x_i) + \delta_i f^{(2m)}(x_i)),$$

and it is precise for every $f \in \pi_{2n+2m-1}$. Moreover

$$d_j = \xi c_j(m) + (1-\xi)\, b_j(0, m, n), \qquad j = 0, ..., m-1$$

$$\gamma_i = (1-\xi)\, \lambda_i(0, m, n), \qquad i = 1, ..., n,$$

$$\delta_i = \xi \frac{(-1)^m}{(2m)!} \lambda_i(m, 0, n), \qquad i = 1, ..., n.$$

It remains to prove that there exists $\xi \in \mathbb{R}$ such that

(i) $Q(f; \xi; m, 0)$ is exact for every $f \in \pi_{2n+2m+1}$.

(ii) $d_{m-1} = 0$.

In order to establish (i) let us consider the Birkhoff interpolation problem based on the information

$$\begin{aligned} & & f^{(j)}(\pm 1) \quad & \text{for} \quad j = 0, ..., m-1, \\ f^{(2m)}(x_i) \quad & \text{and} \quad & f^{(2m+1)}(x_i) \quad & \text{for} \quad i = 1, ..., n. \end{aligned} \tag{17}$$

It follows from the theorem of Atkinson and Sharma [2] that this problem is regular. Therefore there exists a unique polynomial Ω of degree $2n + 2m$ with leading coefficient one which satisfies

$$\Omega^{(j)}(\pm 1) = 0, \qquad j = 0, ..., m-1,$$

$$\Omega^{(2m)}(x_i) = \Omega^{(2m+1)}(x_i) = 0, \qquad i = 1, ..., n.$$

$\Omega(x)$ does not vanish for $x \in (-1, 1)$. Indeed, suppose that there exists $y \in (-1, 1)$ for which $\Omega(y) = 0$. This implies that the Birkhoff interpolation problems determined by the information $f(y)$ and (17) is singular, which contradicts the theorem of Atkinson and Sharma. It is not difficult to see that $\operatorname{sign} \Omega(x) = (-1)^m$ for $x \in (-1, 1)$. Therefore $Q_2(\Omega; m, 0) = 0$ and $Q_1(\Omega; 0, m) \neq 0$, because the coefficients λ_i of Q_1 are all positive. Thus if

$$\xi = 1 - \left(\int_{-1}^{1} \Omega(x)\, dx / Q_1(\Omega; 0, m) \right)$$

then $Q(f; \xi; m, 0)$ is precise for Ω. It remains to note that Q is exact for odd polynomials because of symmetry.

It is easily seen that (ii) is satisfied if one takes $\xi = (1 - \binom{n+2m}{n})^{-1}$. ∎

Proof of Theorem 2. It suffices to find a polynomial h of degree less than $2n + 2m$ for which

$$\begin{aligned} h^{(j)}(\pm 1) &= 0 \quad \text{for} \quad j = 0, ..., m-1, \\ h(x_i) &= 0 \quad \text{for} \quad i = 1, ..., n, \\ \text{and} \quad h^{(2m)}(x_i) &= 0 \quad \text{for} \quad i = 1, ..., n. \end{aligned} \tag{18}$$

$h_{mn}(x) := (1-x^2)^m P_n^{(m)}(x)$ is such a polynomial. Indeed, obviously it satisfies (18). On the other hand, in view of (7) and (8) we have

$$\begin{aligned}\left(\frac{d}{dx}\right)^{2m} &[(1-x^2)^m P_n^{(m)}(x)] \\ &= \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^{n+2m} [(1-x^2)^{m+n}] \\ &= (-1)^m 2^m \frac{(n+m)!}{n!}\left(\frac{d}{dx}\right)^m P_{n+m}(x) \\ &= (-1)^m \frac{(n+2m)!}{n!} P_n^{(m)}(x). \quad \blacksquare \qquad (19)\end{aligned}$$

Proof of Theorem 3. The incidence matrix E of the $(0, 2m)$ interpolation problem with $2m$ nodes is poised since E is a decomposable matrix (see [5, Theorem 1.4; 2]). It follows from Theorem A that the interpolatory q.f. is exact for every polynomial of degree $4m-1$. Obviously polynomial $h_{m,2m}(x) := (1-x^2)^m P_{2m}^{(m)}(x)$ belongs to π_{4m} and has a nonzero leading coefficient. Applying identity (19) for $n=2m$ we obtain $Q_{0,2m}(h_{m,2m})=0$. On the other hand, the orthogonality of $P_{2m}^{(m)}(x)$ implies $\int_{-1}^{1} h_{m,2m}(x)\,dx=0$. Thus $Q_{0,2m}$ is exact for $h_{m,2m}$. It is exact also for the odd polynomials. ∎

4. Applications and Remarks

Let us state formulae (3) and (4) for $m=1$ and $m=2$. For $m=1$ formula (3) reduces to the quadrature formula obtained by Varma in [13, 14], and (4) to the formula constructed by the author in [3]. For $m=2$, (3) takes the form

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx \approx &\frac{2(32n^3+224n^2+479n+345)}{3(n+3)(n+4)(2n+5)(2n^2+5n+5)}(f(-1)+f(1)) \\ &+\frac{6}{n(2n+5)(2n^2+5n+5)}(f'(-1)-f'(1)) \\ &+\frac{(n+1)(n+2)}{n(n+3)(n+4)(2n+5)(2n^2+5n+5)} \\ &\times \sum_{i=1}^{n} \frac{2(2n+1)}{(1-x_i^2)^3\,[P_n^{(2)}(x_i)]^2} \\ &\times \{2^5(n+1)(n+2)(2n+1)\,2n+3)\,f(x_i) \\ &+8(1-x_i^2)^2 f^{(\mathrm{IV})}(x_i)\}, \qquad (20)\end{aligned}$$

where $\{x_i\}_1^n$ are the zeros of $P_n^{(2)}$ and (20) has ADP $= 2n+5$. This is very similar to a q.f. of Varma and Saxena [15, Theorem 2]. Let us note, however, that (20) has a higher ADP because of the proper choice of the nodes. For $m=2$, (4) takes the form

$$\int_{-1}^{1} f(x)\,dx \approx \frac{16}{3(n^2+5n+10)}\,(f(-1)+f(1))$$
$$+\frac{2^5(n+1)(n+2)}{n(n+5)(n^2+5n+10)} \sum_{i=1}^{n} \frac{1}{(1-x_i^2)^3\,[P_n^{(2)\prime}(x_i)]^2}$$
$$\times\left\{(n+1)(n+2)\,f(x_i)+\frac{(1-x_i^2)^2}{(n+3)(n+4)}\,f^{(\mathrm{IV})}(x_i)\right\}, \quad (21)$$

where $\{x_i\}_1^n$ are the zeros of $P_n^{(2)}$ and (21) has ADP $= 2n+3$. This can be considered as a (0, 4) q.f. with set of nodes $\mathbf{X}_n^{(2)} \cup \{1, 1\}$, i.e., the zeros of $(1-x^2)\,P''_{n+2}(x)$. Moreover it is exact for polynomials of degree $2n+3$. Note that it is not known whether the corresponding interpolation problem is regular but if it is, then Theorem A shows that the interpolatory q.f. coincides with (21). For $m=3$, (4) takes the form

$$\int_{-1}^{1} f(x)\,dx \approx \left(\binom{n+6}{n}-1\right)^{-1} \frac{n(n+7)}{12}\,(f(-1)+f(1)) + \left(\binom{n+6}{n}-1\right)^{-1}$$
$$\times \frac{13n^4+182n^3+873n^2+1852n-1160}{2400}\,(f'(-1)-f'(1))$$
$$+\left\{1-\left(1-\binom{n+6}{n}\right)^{-1}\right\} \sum_{i=1}^{n} \lambda_{in}^{(3)}(1-x_i^2)^{-3} f(x_i)$$
$$+\left\{6!\cdot\left(1-\binom{n+6}{n}\right)\right\}^{-1} \sum_{i=1}^{n} \lambda_{in}^{(3)} f^{(6)}(x_i),$$

where $\{x_i\}_1^n$ are the zeros of $P_n^{(3)}$ and this formula has ADP $= 2n+5$.

Applying the result in Lemma 4 for $m=L=1$, $n \in \mathbb{N}$, and $\xi = -8(2n^2+10n+9)/n(n+5)(3n^2+15n+14)$ we obtain the formula

$$\int_{-1}^{1} f(x)\,dx \approx 24(n(n+5)(3n^2+15n+14))^{-1}\,(f'(-1)-f'(1))$$
$$+\,64(2n^2+10n+9)(n(n+5)(3n^2+15n+14)\cdot(n+1)_4)^{-1}$$
$$\times\,(f''(-1)+f''(1))+3(n+1)_4(n(n+5)(3n^2+15n+14))^{-1}$$
$$\times \sum_{i=1}^{n} \lambda_{in}^{(2)}(1-x_i^2)^{-2} f(x_i)$$
$$+\,4(2n^2+10n+9)(n(n+5)(3n^2+15n+14))^{-1}$$
$$\times \sum_{i=1}^{n} \lambda_{in}^{(2)}(1-x_i^2)^{-1} f''(x_i), \quad (22)$$

where $\{x_i\}_1^n$ are the zeros of $P_n^{(2)}$ and (22) is exact for every polynomial of degree $2n+3$. Note again that Theorem A implies that if the Birkhoff interpolation problem described by the information involved in this formula is regular, then (22) coincides with the corresponding interpolatory q.f.

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